

Quantum Bound States Around Black Holes

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Quantum mechanics around black holes has shown to be one of the most fascinating fields of theoretical physics. It involves both general relativity and particle physics, opening new eras to establish the groundings of unified theories. In this article, we show that quantum bound states with no classical equivalent – as it can easily be seen at the dominant monopolar order – should be formed around black holes for massive scalar particles. We qualitatively investigate some important physical consequences, in particular for the Hawking evaporation mechanism and the associated greybody factors.

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The investigation of quantum bound states around black holes, with important consequences both for the dynamics of particles and for the Hawking evaporation process, requires to solve relativistic quantum mechanical equations in a curved background while taking into account a non-vanishing mass. The Klein-Gordon equation of motion for a scalar field Φ with mass μ in a space-time with metric $g_{\alpha\beta}$ can be expressed as :

$$\frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta\Phi) + \mu^2\Phi = 0. \quad (1)$$

Writing $\Phi = e^{-i\omega t}Y_m^\ell(\theta, \varphi)R(r)$ to split the temporal, angular and radial parts of the field (where Y_m^ℓ are the spherical harmonics), the radial function $R(r)$ obeys, in a 4-dimensional Schwarzschild background :

$$\left[\frac{h(r)}{r^2} \frac{d}{dr} h(r) r^2 \frac{d}{dr} + \omega^2 - h(r) \left(\frac{\ell(\ell+1)}{r^2} + \mu^2 \right) \right] R(r) = 0, \quad (2)$$

where $h(r)$ is defined by the metric $ds^2 = h(r)dt^2 - dr^2/h(r) - r^2d\Omega^2$ (see, e.g., [1] and references therein for general techniques associated with quantum fields in a Schwarzschild spacetime). Performing the change of variables $r \rightarrow r_*$ and $R(r) \rightarrow U(r)$ where r_* is the tortoise coordinate such that $dr_* = dr/h(r)$ and $U(r) = rR(r)$, this equation takes a Schrödinger-like form

$$\frac{d^2U}{dr_*^2} + (\omega^2 - V_\ell^2(r))U = 0, \quad (3)$$

with a potential

$$V_\ell^2(r) = \left(1 - \frac{r_H}{r}\right) \left(\frac{\ell(\ell+1)}{r^2} + \frac{r_H}{r^3} + \mu^2 \right), \quad (4)$$

where r_H stands for the Schwarzschild radius and ℓ for the angular quantum number. This allows the usual

quantum mechanical techniques to be employed in the tortoise coordinate system. Through out this article, we use the Chandrasekhar convention : the last term of equation 3 is interpreted as the squared potential so as to recover the standard Hamilton-Jacobi equation. On Fig. 1, $V_\ell^2(r)$ is shown for three different values of the mass ($\mu = \{0, \sqrt{0.1}, \sqrt{0.4}\} [r_H^{-1}]$) and two values of the angular momentum ($\ell = 0$ and $\ell = 1$). Depending on μ and ℓ , it can be seen that a local minimum, leading to a bound state, eventually appears. The existence of a potential well depends on the roots of the algebraic equation $\frac{dV_\ell^2}{dr} = 0$:

$$r_H\mu^2r^3 - 2\ell(\ell+1)r^2 - 3r_H(1 - \ell(\ell+1))r + 4r_H^2 = 0. \quad (5)$$

Two roots above r_H exist if the mass μ is lower than a critical value $\mu_+(\ell)$ given by

$$\mu_+^2 = \frac{1}{216r_H^2} \left(-27J_1 + \sqrt{729J_1^2 + 432J_2} \right), \quad (6)$$

with

$$J_1 = \ell^3(\ell+1)^3 + \ell^2(\ell+1)^2 - \ell(\ell+1) - 1, \quad (7)$$

$$J_2 = \ell^2(\ell+1)^2 [9 + 14\ell(\ell+1) + 9\ell^2(\ell+1)^2]. \quad (8)$$

In the monopolar case, it takes the simple value $\mu_+ = \frac{1}{2}r_H^{-1}$. This allows to draw the general behavior of the potential. For example, if a particle has a mass $\mu \in [\mu_+(\ell_1), \mu_+(\ell_1+1)]$, then the potential is an increasing bijection from $[r_H, \infty[$ to $[0, \mu[$ for every multipolar order lower or equal to ℓ_1 whereas a potential barrier and a local well appear for multipolar orders strictly higher than ℓ_1 .

The detailed shape of the potential is also determined by another critical mass, hereafter called $\mu_-(\ell)$, which defines the relative height of the potential barrier close to the horizon when compared with the mass of the particle. If the mass of the particle is higher than the barrier, there is no more turning point for the system. The mass is the highest value of the potential if the equation $V_\ell^2(r) = \mu^2$ has no root above r_H . The cubic form

$$r^3 - \frac{\ell(\ell+1)}{r_H\mu^2}r^2 - \frac{1 - \ell(\ell+1)}{\mu^2}r + \frac{r_H}{\mu^2} = 0, \quad (9)$$

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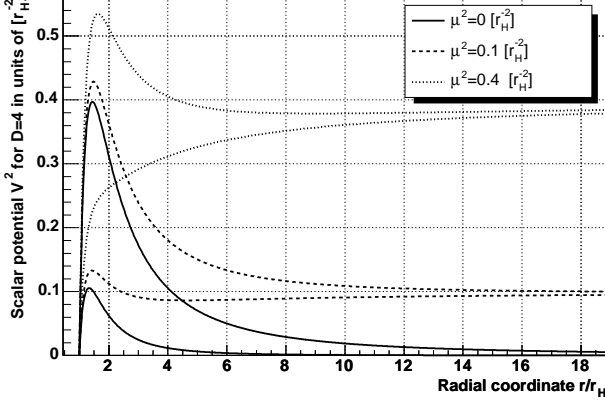


FIG. 1: Square of the potential as a function of the radial coordinate for three values of the mass μ and two values of the angular quantum number ℓ (in each case, $\ell = 0$ for the lower curve and $\ell = 1$ for the upper curve).

satisfies this criterion if $\mu > \mu_-(\ell)$ which is then given by :

$$\mu_-^2 = \frac{1}{27r_H^2} \left(-L_1 + \sqrt{L_1^2 + 27L_2} \right), \quad (10)$$

with

$$L_1 = 2\ell^3(\ell+1)^3 + 3\ell^2(\ell+1)^2 - 3\ell(\ell+1) - 2, \quad (11)$$

$$L_2 = \ell^2(\ell+1)^2 [\ell(\ell+1) + 1]^2. \quad (12)$$

In the monopolar case, it becomes $\mu_- = \frac{2}{\sqrt{27}} r_H^{-1}$.

The potential is always zero at $r = r_H$ and tends to μ for $r \rightarrow \infty$. If i) $\mu < \mu_-(\ell)$, the potential reaches a maximum higher than μ and then reaches a minimum, if ii) $\mu = \mu_-(\ell)$, the maximum is exactly equal to μ and a minimum also appears, if iii) $\mu_-(\ell) < \mu < \mu_+(\ell)$ the potential reaches a maximum lower than μ and still admits a minimum whereas if iv) $\mu \geq \mu_+(\ell)$ the potential is a monotonically increasing function of r . This behavior is illustrated for $\ell = 0$ on Fig. 2 for masses corresponding to those four specific cases. It can be seen that bound states –due to the local minimum– can appear at the monopolar order, with *no classical equivalent*. Although particles can be classically trapped around a black hole, no such state can be found without angular momentum if the quantum behavior is not taken into account. Furthermore, whatever the mass of the particle, a bound state will appear for high enough multipolar orders so that $\mu < \mu_+(\ell)$. This makes this phenomenon of "particle trapping" quite generic. Those bound states are described by quasi-stationary quantum states which cannot reach spatial infinity but can still transit to the black hole by tunneling back the gravitational barrier.

As the qualitative features can easily be understood at the monopolar order, this particular value of the quantum angular momentum is now assumed. When $\mu < \mu_+$,

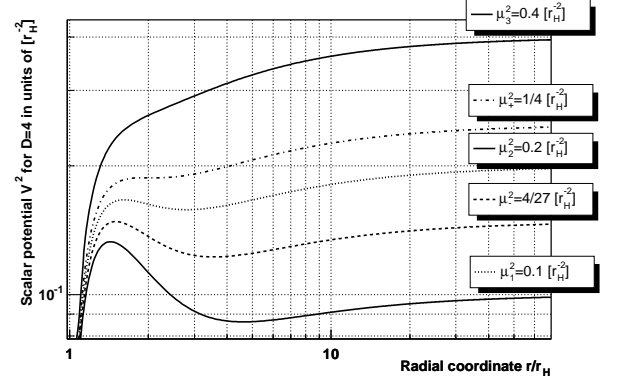


FIG. 2: Squared monopolar potential for five particle masses : i) $\mu^2 = 0.1 r_H^{-2}$ corresponding to $\mu < \mu_-(0)$, ii) $\mu = \mu_-(0)$, iii) $\mu^2 = 0.2 r_H^{-2}$ corresponding to $\mu_-(0) < \mu < \mu_+(0)$, iv) $\mu = \mu_+(0)$ and v) $\mu^2 = 0.4 r_H^{-2}$ corresponding to $\mu > \mu_+(0)$.

the positions of the potential barrier (r_-) and of the local minimum (r_+) can be analytically determined as :

$$r_{\pm} = \frac{1}{\mu} \left[\cos(\theta) \pm \sqrt{3} \sin(\theta) \right], \quad (13)$$

with

$$\theta = \frac{1}{3} \arctan \left[\sqrt{\left(\frac{\mu_+}{\mu} \right)^2 - 1} \right]. \quad (14)$$

The asymptotic behavior is in agreement with the monopolar potential for a massless particle :

$$\lim_{\mu \rightarrow 0} r_+ = +\infty, \quad \lim_{\mu \rightarrow 0} r_- = \frac{4}{3} r_H, \quad \lim_{\mu \rightarrow \mu_+} r_{\pm} = 2r_H. \quad (15)$$

The latter case, $\mu \rightarrow \mu_+$, correspond to the appearance of a saddle point in $r = 2r_H$ which represents the degeneracy of the maximum and minimum of the potential.

particle (mass)	$\mu = \mu_-$ for $\ell = 0$		$\mu = \mu_+$ for $\ell = 0$	
	M_{BH} [kg]	T_H [MeV]	M_{BH} [kg]	T_H [MeV]
electron (511 keV)	9×10^{13}	0.1	1.2×10^{14}	0.08
charm (1.3 GeV)	4×10^{10}	248	5×10^{10}	190
W boson (80 GeV)	6×10^8	16×10^3	8×10^8	12×10^3

TABLE I: Masses and temperatures expected for a black hole to have the critical masses μ_{\pm} close to the masses of some *standard model* particles.

Clearly, the mass has to be close to the critical values μ_+ and μ_- if its consequences on the propagation are to be substantial otherwise either no bound state exist

or the particle will be ultra-relativistic. In this case, the intricate shape of the potential leads to new effects due both to the event horizon and to the mass of the quantum which is usually neglected when considering the propagation of fields in the vicinity of a black hole. When the potential exhibits a minimum, bound states appear with an energy lower than the mass of the particle, due to the gravitational binding energy. As this can even happen for $\ell = 0$, a spherical halo of quanta "orbiting" the black hole can be expected. Furthermore, if the mass is between $\mu_+(\ell - 1)$ and $\mu_+(\ell)$, bound states will exhibit an angular distribution dominated by the lowest multipolar order allowing for a minimum in the potential, that is with a distribution roughly given by $Y_m^\ell(\theta, \varphi)$. So as to fix the orders of magnitude, table I gives the masses and temperatures black holes should have so that the effects studied in this article become important for some *standard model* particles. Although they are not spinless –therefore requiring to investigate the master equation for fermions and gauge bosons [2]– the main qualitative features can safely be inferred from the scalar case. Figure 3 displays a summary of the expected behaviors.

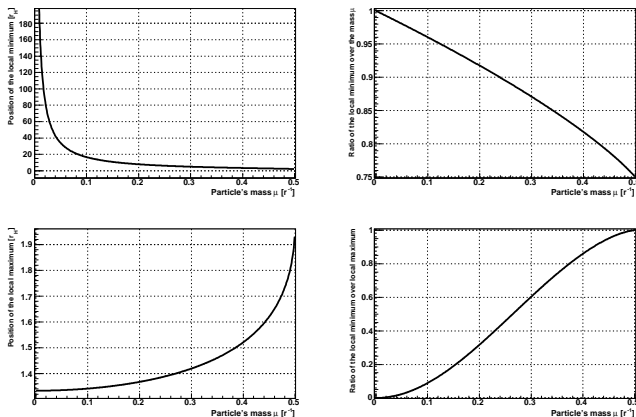


FIG. 3: Positions of the extrema of the potential as a function of the mass (left) and ratios of the minimum over the mass and over the maximum (right).

The existence of bound states will play an important role in the Hawking evaporation mechanism which was initially described in [3] without taking into account this phenomenon.

First, some low-energy particles will be trapped by the potential well. They won't reach infinity and the spectrum will be modified in a way quite similar to what could happen due to a QCD halo [4]. When the black hole evaporates, new bound states will appear each time the temperature becomes of the same order than the critical masses μ_\pm associated with existing particles. It can be easily checked that those states do indeed exist by comparing the depth of the potential with the zero-point energy. Assuming that the potential shape is dominated by the second order term, the dynamics

is the one of a harmonic oscillator with a frequency $\tilde{\omega}$ given by the curvature of the potential around its minimum: $\tilde{\omega} = \sqrt{d^2V^2/dr_\star^2}|_{r_+}$. The curvature of the potential must be evaluated as a function of the tortoise coordinate because this is the coordinates system where the radial part of the Klein-Gordon equation is of the Schrödinger type. The zero-point energy is simply given by $\omega_0 = \frac{\tilde{\omega}}{2}$. This approximated expression is plotted on figure 4 alongside the mass and the maximum of the potential as functions of the mass of the particle. Each time a minimum does appear, the zero-point energy remains smaller than the mass and the gravitational potential barrier, allowing bound states to exist.

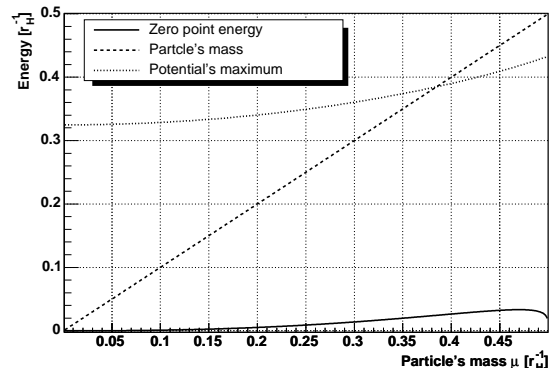


FIG. 4: zero-point energy, mass and maximum of the potential as a function of the mass of the particle in units of r_H^{-1} .

The normal frequencies and bandwidths of the bound states can be evaluated at the semi-classical order using the techniques developed in [5]. The spectrum will be infinite if $\mu < \mu_-$ and finite if $\mu_- < \mu < \mu_+$. The Bohr-Sommerfeld rule, whose validity in a relativistic framework was established in [5], reads as $\forall n \in \mathbb{N}$, $\int_{r_1(\omega)}^{r_2(\omega)} \sqrt{\omega^2 - V^2(r)} \frac{dr}{h(r)} = \left(n + \frac{1}{2}\right) \pi$, the highest frequency allowed for a bound state being μ for $\mu < \mu_-$ and $V(r_-)$ for $\mu > \mu_-$. In the later case, the left-hand side integral is clearly finished and there exist n_{max} states. If $\mu < \mu_-$, the upper bound of the integral is infinite when $\omega = \mu$ and, the function to be integrated being proportional to $r^{-1/2}$ near $+\infty$, the spectrum is expected to be infinite. With the appropriate change of variables in the cubic equation giving the turning points, it can be shown that the normal frequency spectrum of the resonances depends only on μr_H and can be written as $\{\omega_n\}_{n \in \mathbb{N}} = \left\{ \frac{f_n(\mu r_H)}{r_H} \right\}_{n \in \mathbb{N}}$. Table II gives some normal frequencies and the associated bandwidths, as numerically obtained at the semi-classical order, following [6] to evaluate the tunnel probability.

Then, the mass of the particle will also drastically modify the greybody factors that account for the non-trivial part (gravitational barrier and centrifugal potential) of

$\mu = 0.25 \times \mu_+$					
n	0	1	2	3	4
ωr_H	$1.24.10^{-1}$	$1.25.10^{-1}$	$1.25.10^{-1}$	$1.25.10^{-1}$	$1.25.10^{-1}$
Γr_H	$1.49.10^{-2}$	$1.49.10^{-2}$	$1.50.10^{-2}$	$1.50.10^{-2}$	$1.50.10^{-2}$
$\mu = \mu_-$					
n	0	1	2	3	4
ωr_H	$3.75.10^{-1}$	$3.83.10^{-1}$	$3.84.10^{-1}$	$3.84.10^{-1}$	$3.85.10^{-1}$
Γr_H	$1.68.10^{-1}$	$1.87.10^{-1}$	$1.90.10^{-1}$	$1.91.10^{-1}$	$1.92.10^{-1}$
$\mu = 0.8 \times \mu_+$					
n	0	1	2	3	4
ωr_H	$3.88.10^{-1}$	/	/	/	/
Γr_H	$1.91.10^{-1}$	/	/	/	/

TABLE II: Spectrum of the normal frequencies ω and bandwidths Γ evaluated at the WKB order in units of r_H^{-1} for three different masses : $\mu < \mu_-$, $\mu = \mu_-$ et $\mu > \mu_-$.

the couplings between quantum fields and evaporating black holes. The greybody factors (whose detailed study began with [7, 8, 9]) have recently been computed in quite a lot of interesting situations : extra-dimensions [10], de-Sitter spacetime [11], rotating black holes [12, 13], Gauss-Bonnet gravity [14, 15] etc., but up to now the masses of the emitted particles have mostly been ignored (although some good estimates were obtained in [9, 16]). Figure 5 displays the absorption cross section numerically computed by solving the Klein-Gordon equation to evaluate the ingoing and outgoing amplitudes of the wave function at the horizon and at spatial infinity (see, *e.g.*, [11] for a detailed description of the method we have developped). It should be noticed that, when the wavelength of the particle becomes infinite, the cross section diverges, leading to an experimentally relevant enhancement of soft quanta. It can indeed be expressed as $\sigma_g(\omega) = \sum_{\ell} \frac{\pi(2\ell+1)}{k^2} |A_{\ell}|^2$, where k is the momentum (so that $\omega^2 = k^2 + \mu^2$ at spatial infinity) and $|A_{\ell}|^2$ is the transmission coefficient. If the particle has a non-vanishing mass, the potential barrier is not anymore infinitely thick when $\omega \rightarrow \mu$ and the transmission coefficient therefore tends to a finite value. In Fig 6, the flux at infinity emitted by a black hole is displayed when the mass of the emitted quanta are taken into account. As it can be seen, this substantially modifies the usual picture both because of the intrinsic cutoff imposed by the mass and because of more subtle effects included in this analysis, like the selection induced on the allowed quantum

multipolar orders of the outgoing particle.

This study establishes the existence of new bound states around black holes which, at least at the dominant monopolar order, have no classical equivalent. This opens new perspectives to investigate the detailed features of the Hawking spectrum (with possible cosmolog-

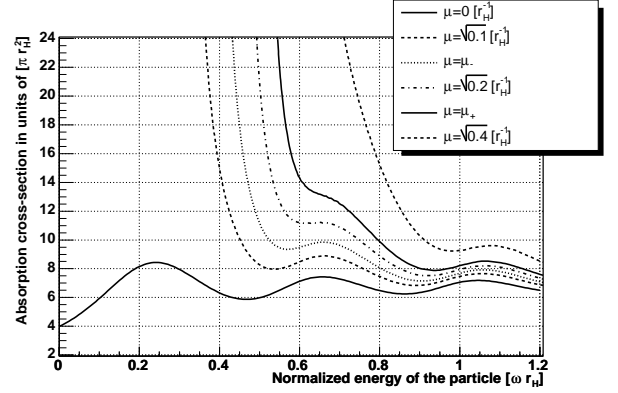


FIG. 5: Absorption cross section for massive scalar particles in units of πr_H^2 as a function of the energy measured at infinity.

ical consequences related, *e.g.*, to the primordial power spectrum -see [17] for recent limits and [18] for a review), the intricate shape of the greybody factors and the propagation of massive quantum fields in the vicinity of a black hole. Not only will the phenomenology change as the spectra should be quantitatively modified but fruitful thought experiments associated with light black holes should also take into account those states.

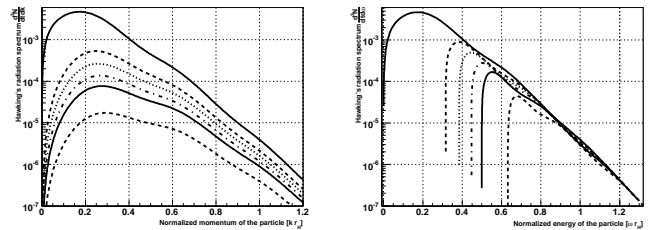


FIG. 6: Flux at infinity emitted by a black hole when taking into account the mass of the emitted particle, displayed on the left as a function of the momentum and on the right as a function of the energy. The different curves share the same convention as in Fig. 6 : $\mu = 0, \sqrt{0.1}, \mu_-, \sqrt{0.2}, \mu_+, \sqrt{0.4} [r_h^{-1}]$ from top to bottom.

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